Pseudopotentials, Lax pairs and Backlund transformations for some variable coefficient nonlinear equations



1991 J. Phys. A: Math. Gen. 24 L513

(http://iopscience.iop.org/0305-4470/24/10/003)

J. Phys. A: Math. Gen. 24 (1991) L513–L518. Printed in the UK

LETTER TO THE EDITOR

# Pseudopotentials, Lax pairs and Bäcklund transformations for some variable coefficient nonlinear equations

Sen-yue Lou

Institute of Modern Physics and Department of Physics, Ningbo Normal College, Ningbo 315211, People's Republic of China

Received 31 October 1991

**Abstract.** Using the method developed by Nucci, we have obtained the pseudopotentials, Lax pairs, Bäcklund transformations and the singularity manifold equations of some variable coefficient nonlinear equations. These equations are generalized Korteweg–de Vries, generalized modified Korteweg–de Vries and generalized Boussinesq equations.

Nucci (1988) showed that if the equations satisfied by the pseudopotential (Wahlquist and Estabrook 1975) are of Riccati type, then one can easily obtain both the Lax equations and the auto-Bäcklund transformation (auto-BT) for the corresponding nonlinear equations. Furthermore, if a pseudopotential $u = u(x, t)$ has the form

$$u_x = ku^2 + F_1(q)u + F_0(q) \tag{1a}$$

$$u_t = G(u, q, q_x, q_{xx}, \ldots) \tag{1b}$$

with $k$ a constant and $G$ a polynomial of second order in $u$, then the Lax equations with dependent variable $\psi(x, t)$ and the singularity manifold equations with dependent variable $\phi(x, t)$ can be obtained immediately by the transformations

$$u = -\frac{1}{k}(\ln \psi)_x \tag{2}$$

from (1) and

$$\phi = \psi_1/\psi_2 \tag{3}$$

from the Lax equations respectively (Nucci 1989). $\psi_1$ and $\psi_2$ in (3) are two linear independent solutions of the Lax equations.

In this letter, we will apply the method to three variable coefficient nonlinear equations. These equations are as follows.

(i) The generalized Korteweg–de Vries equation (GKdVE)

$$q_t = q_{xxx} + 6qq_x + (Fx + G)q_x + 2Fq \tag{4}$$

with $F$ and $G$ arbitrary functions of $t$. When $F$ and $G$ tend to zero, (4) reduces back to the well known KdV equation; this is why we refer to (4) as the GKdVE. Equation (4) can be derived both from the similarity reductions of the Kadomtsev–Petviashvili equation (KPE) (Lou *et al* 1991) and from the variable coefficient and non-isospectral KdVE (Chan and Li 1989) (by taking $K_0 = -1$, $K_1 = -G$ and $F = h$). The soliton solutions of (4) with time varying boundary conditions have been also studied by Chan and Li (1989).

(ii) The generalized modified Korteweg-de Vries equation (GMKdVE)

$$q_t = q_g x_{xx} - 6q^2 q_x + (Fx + G) q_x + Fq \tag{5}$$

with $F$ and $G$ any two functions of $t$. Equation (5) can be obtained by a generalized Miura transformation

$$q'_x = -q'^2 - q \tag{6a}$$

$$q'_t = -(2q + Fx + G) q'^2 + (2q_x + F) q' - q(2q + Fx + G) - q_{xx} \tag{6b}$$

where $q$ and $q'$ satisfy the GKdVE (4) and GMKdVE (5) respectively. When $F = 0$, $g = 0$, (5) reduces to the MKdVE and (6) reduces to a known Miura transformation.

(iii) The generalized Boussinesq equation (GBE)

$$\begin{aligned} q_{xxxx} + 6q_x^2 + 6qq_{xx} + q_{tt} + (2F_1x + F_2) q_{xt} + 6F_1 q_t + \tfrac{1}{4}(2F_1 x + F_2)^2 q_{xx} \\ + ((7F_1^2 + F_{1t}) x + \tfrac{7}{2} F_1 F_2 + \tfrac{1}{2} F_{2t}) q_x + (8F_1^2 + 2F_{1t}) q = 0 \end{aligned} \tag{7}$$

with $F_1$ and $F_2$ two arbitrary functions of $t$. Equation (7) can also be obtained from the similarity reductions of the KPE. In Lou *et al* (1991), by means of the similarity reductions of the KPE, we have reported two nonlinear equations which possess different forms from (4) and (7) but they are equivalent.

Following the method developed by Nucci (1988, 1989), we assume there exists a pseudopotential $u = u(x, t)$ which has the form

$$u_x = ku^2 + F_1(q, x, t) u + F_0(q, x, t) \tag{8a}$$

$$u_t = G(u, q, x, t, q_x, q_{xx} \ldots) \tag{8b}$$

with $G$ a polynomial of second order in $u$. Due to the coefficients of (4) being variable, the independent variables $(x, t)$ appear explicitly in (8).

Starting from the integrability condition of (8), $u_{xt} = u_{tx}$, when (4) is true, we get a simple solution

$$u_x = ku^2 + k^{-1}(q - \lambda) \tag{9a}$$

$$u_t = k(2q + Fx + 4\lambda + G) u^2 + (2q_x + F) u + k^{-1}(q - \lambda)(2q + Fx + 4\lambda + G) - q_{xx}. \tag{9b}$$

For simplicity, in the following calculation we take the spectral parameter $\lambda = 0$ and the constant $k = -1$.

Now the Lax equations of the GKdVE (4) can be obtained easily by means of the transformations (2) with $k = -1$. The result reads

$$\psi_{xx} = -q\psi \tag{10a}$$

$$\psi_t = (2q + Fx + G)\psi_x - (q_x + \tfrac{1}{2}F)\psi. \tag{10b}$$

Alternatively, if we introduce the linearizing transformation

$$u = v_1 / v_2 \tag{11}$$

instead of (2), we obtain the following Lax equations in the AKNS form (Ablowitz and Segur 1981):

$$V_x = AV \tag{12a}$$

$$V_t = BV \tag{12b}$$

where

$$V=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad A=\begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} \tag{13, 14}$$

and

$$B=\begin{pmatrix} q_x+\frac{1}{2}F & -q(2q+Fx+G)-q_{xx} \\ 2q+Fx+G & -(q_x+\frac{1}{2}F) \end{pmatrix}. \tag{15}$$

The Bäcklund transformation (BT) of the GKdVE can also be obtained easily from the pseudopotential $u$ shown by (9). Another solution $q^*$ of (4) will have a pseudopotential $u^*$ such that from (9$a$) with $k=-1$

$$u_x^* = -u^{*2} - q^*. \tag{16}$$

If we suppose

$$u^* = -u \tag{17}$$

then (16) becomes

$$u_x = u^2 + q^*. \tag{18}$$

Combining (9$a$) and (18) leads to

$$q+q^* = -\frac{1}{2}\left(\int (q-q^*)\,\mathrm{d}x\right)^2 \tag{19}$$

the spatial part of the auto-BT; the temporal part of the auto-BT can be obtained similarly but we omit it here.

Finally, because the pseudopotential of the GKdVE has the form (9$a$), the singularity manifold equation with dependent variable $\phi=\phi(x,t)$ can be obtained both by the transformation (3) through the Lax equations (10) and by the transformation

$$u = -\tfrac{1}{2}(\ln \phi_x)_x \tag{20}$$

through the pseudopotential equation (9) directly. The result reads

$$\phi_t/\phi_x - \{\phi; x\} = Fx+G \tag{21$a$}$$

where the spectral parameter has been taken as zero and

$$\{\phi; x\} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{21$b$}$$

is the Schwartz derivative. Both (21$b$) and (21$a$) are invariant under a Möbius transformation (Weiss 1983). When $F=0$ and $G=0$, (21) is just the singularity manifold equation of the KdVE as expected.

As in the GKdVE case, it is easy to obtain that the pseudopotential equations of GMKdVE (5) has the form

$$u_x = u^2 - 2uq \tag{22$a$}$$

$$u_t = (-2uq_x - 2uq^2 + (Fx+G)u)_x \tag{22$b$}$$

It is straightforward to get the Lax equations from (20) by transformation (2) with $k=1$:

$$\psi_{xx} = -2q\psi_x \tag{23$a$}$$

$$\psi_t = -2(q_x + q^2 - 2Fx - 2G)\psi_x \tag{23$b$}$$

or in the following AKNS system form:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} -q & 0 \\ -1 & q \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{24a}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} -q_{xx}+2q^3-2q(Fx+G)+\frac{1}{2}F & 0 \\ 2q_x+2q^2-Fx-G & q_{xx}-2q^3+2q(Fx+G)-\frac{1}{2}F \end{pmatrix}. \tag{24b}$$

Because there is no pseudopotential such that

$$u_x = ku^2 + F_0(q, x, t) \tag{25}$$

with $k$ constant is true, the singularity manifold equation of the GMKdVE cannot be obtained by the transformation (20) but can be obtained by the trivial spectral function itself ($\phi = \psi$). That is to say, the singularity manifold equation of the GMKdVE can be obtained by cancelling $q$ in (23) directly:

$$\frac{\psi t}{\psi x} - \{\psi; x\} = 2Fx + 2G. \tag{26}$$

It is worth mentioning that the singularity manifold equation (26) of the GMKdVE and that of the GKdVE possess the same form only for $F = 0$ and $G = 0$ even if the spectral parameter has been taken as zero.

The Bäcklund transformation of the GMKdVE can also be obtained from pseudopotential equation (22) by assuming another solution $q^*$ of the GMKdVE has a pseudopotential $u^*$

$$u^*_x = u^{*2} - 2u^* q^*. \tag{27}$$

By requiring that $u^* = -u$ yields the following spatial part of an auto-BT of the GMKdVE:

$$q - q^* = \exp\left\{ -\int (q + q^*)\, dx \right\}. \tag{28}$$

As in the MKdVE case (Nucci 1989), an alternative pseudopotential can be given by

$$\hat{u}_x = \hat{u}^2 + q_x - q^2 \tag{29a}$$

$$\hat{u}_t = (2(q_x - q^2)\hat{u} + q_{xx} - 2qq_x + (Fx + G)\hat{u})_x. \tag{29b}$$

It is quite interesting that the pseudopotential $\hat{u}(x, t)$ shown by (29) is also a solution of the GMKdVE; in other words the pseudopotential equation (29) is also the auto-BT equation of the GMKdVE at the same time. The singularity manifold equation (26) of the GMKdVE can now be obtained by transformation (20) (with $u \to \hat{u}$, $\phi \to \psi$) thanks to the fact that the pseudopotential has the form (29). Actually, (20) is the same as (23$a$) owing to the fact that $\hat{u}$ is really a solution of the GMKdVE. Finally, a non-trivial spectral function of (5) can be given by the linearization procedure of the pseudopotential equation (29)

$$\hat{u} = -(\ln \hat{\psi})_x. \tag{30}$$

The final Lax equations read

$$\hat{\psi}_{xx} = -(q_x - q^2)\hat{\psi} \tag{31a}$$

$$\hat{\psi}_t = 2(q_x - q^2 + \tfrac{1}{2}Fx + \tfrac{1}{2}G)\hat{\psi}_x - (q_{xx} - 2qq_x + \tfrac{1}{2}F)\hat{\psi} \tag{31b}$$

or the Lax equations in the AKNS form which are obtained from $\hat{u}=\hat{v}_1/\hat{v}_2$:

$$\begin{pmatrix}\hat{v}_1\\ \hat{v}_2\end{pmatrix}_x=\begin{pmatrix}0 & q_x-q^2\\ 1 & 0\end{pmatrix}\begin{pmatrix}\hat{v}_1\\ \hat{v}_2\end{pmatrix} \tag{30a}$$

and

$$\begin{pmatrix}v_1\\ v_2\end{pmatrix}_t=\begin{pmatrix}q_{xx}-2qq_x+\frac{1}{2}F & 2(q_x-q^2)^2+(Fx+G)(q_x-q^2)+q_{xxx}-2q_x^2-2qq_{xx}\\ -2q_x-2q^2-Fx-G & -q_{xx}+2qq_x-\frac{1}{2}F\end{pmatrix}$$
$$\times\begin{pmatrix}v_1\\ v_2\end{pmatrix}s. \tag{32b}$$

Although the Boussinesq equation (BE) is not an evolution equation, all the properties of an integrable model mentioned above can be obtained by the transformation $t=0$ and $y=\tau$ from the KPE (Nucci 1989). The results of the GBE (7) can be obtained by extending the results of the BE. The pseudopotential reads

$$u_x=u^2+\frac{\sqrt{3}}{3}(d_1x+\tfrac{1}{2}F_2)u+q+\frac{\sqrt{3}}{3}\int^x u_t\,\mathrm{d}x \tag{33a}$$

$$u_t=-\frac{\sqrt{3}}{2}qu+u\int^x u_t\,\mathrm{d}x-F_1u-\frac{\sqrt{3}}{3}(F_1x+\tfrac{1}{2}F_2)\int^x u_t\,\mathrm{d}x-\frac{\sqrt{3}}{3}(F_1x+\tfrac{1}{2}F_2)^2u$$
$$+\frac{3}{4}\int^x(q_t+F_1q)\,\mathrm{d}x-\frac{\sqrt{3}}{4}q_x-\tfrac{1}{4}(F_1x+\tfrac{1}{2}F_2)q. \tag{33b}$$

Substituting the transformation (2) with $k=-1$ into (33), we obtain the Lax equations of the GBE (7):

$$\psi_{xxx}=-\tfrac{3}{2}q\psi_x-\frac{\sqrt{3}}{4}\left[\int^x(q_t+F_1q)\,\mathrm{d}x+\sqrt{3}q_x+(F_1x+\tfrac{1}{2}F_2)q\right]\psi+\lambda\psi \tag{34a}$$

$$\psi_t=\sqrt{3}\psi_{xx}+(-F_1x-\tfrac{1}{2}F_2)\psi_x+\sqrt{3}q\psi \tag{34b}$$

where $\lambda$ is the arbitrary spectral parameter. Although we can linearize the GBE by means of the second-order scattering problem

$$\psi_{xx}=a\psi \tag{35a}$$

$$\psi_t=b\psi_x+c\psi \tag{35b}$$

as in the BE case, it is not possible, however, to introduce an arbitrary spectral parameter $\lambda=a-q$ into (35) even for $F_1=0$ and $F_2=0$ (Weiss 1983).

By means of the transformation (3), we have the singularity manifold equation of the GBE (7):

$$\left[\{\phi;x\}+\frac{1}{2}\left(\frac{\phi_t^2}{\phi_x^2}\right)\right]_x+\left(\frac{\phi_t}{\phi_x}\right)_t+(2F_1x+F_2)\left(\frac{\phi_t}{\phi_x}\right)_x+2F_1\frac{\phi_t}{\phi_x}$$
$$+3F_1(F_1x+\tfrac{1}{2}F_2)+(F_{1t}x+\tfrac{1}{2}F_{2t})=0 \tag{36}$$

which is invariant under a Möbius transformation and will reduce to the singularity manifold equation of the BE when $F_1=0$ and $F_2=0$.

In order to obtain the Bäcklund transformation of the GBE, we write the pseudopotential $u^*$ for another solution of the GBE $q^*$ as

$$u_x^* = u^{*2} - \frac{\sqrt{3}}{3}(F_1 x + \tfrac{1}{2}F_2)u^* + q^* - \frac{\sqrt{3}}{3}\int^x u_t^* \, dx \tag{37a}$$

$$u_t^* = \frac{\sqrt{3}}{2} q^* u^* + u^* \int^x u_t^* \, dx - F_1 u^* + \frac{\sqrt{3}}{3}(F_1 x + \tfrac{1}{2}F_2)\int^x u_t^* \, dx + \frac{\sqrt{3}}{3}(F_1 x + \tfrac{1}{2}F_2)^2 u^* + \frac{3}{4}\int^x (q_t^* + F_1 q^*) \, dx + \frac{\sqrt{3}}{4} q_x^* - \tfrac{1}{4}(F_1 x + \tfrac{1}{2}F_2) q^* \tag{37b}$$

which is equivalent to (33) because the GBE (7) is invariant under $t \to t$, $F_1 \to -F_1$ and $F_2 \to -F_2$.

After substituting $u^* = -u$ into (37$a$) and combining with (33$a$), we have the spatial part of the auto-BT of the GBE immediately:

$$(Q - Q^*)^2 + 2(Q_x + Q_x^*) + \frac{2}{\sqrt{3}}\int (Q_t - Q_t^*) \, dx + \frac{\sqrt{3}}{3}(F_1 x + \tfrac{1}{2}F_2)(Q - Q^*) = 0 \tag{38}$$

with

$$Q = \int q \, dx \qquad \text{and} \qquad Q^* = \int q^* \, dx. \tag{39}$$

Starting from the pseudopotentials which are of Riccati type, we obtain the Lax equations, auto-BT and the singularity manifold equations of three variable coefficient completely integrable nonlinear equations; GKdVE (4), GMKdVE (5) and GBE (7). These equations can be obtained by the similarity reductions of the KPE (equations (4) and (7)) or by a generalized Miura transformation (equation (5)). All the results reduce back to the results of the usual KdVE, MKdVE and BE (Nucci 1989) when the variable coefficients—some arbitrary functions of $t$—tend to zero.

In Weiss *et al* (1983) the Painlevé property for partial differential equations was established. The Lax pairs, auto-BT and singularity manifold equations etc can also be obtained from the Weiss configuration efficiently, but here we do not discuss this method further. There exist various other interesting properties such as infinite conservation quantitites, bilinear forms, prolongation structures etc; we will discuss these properties elsewhere in further investigations of these models.

This work was supported by NSF in China. The author would like to thank Professors G-j Ni, D-f Chen, Doctors G-x Huang, H-y Ruan and W-z Chen for helpful discussions.

## References

Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transformation* (Philadelphia, PA: SIAM)
Chan W L and Li K S 1989 *J. Math. Phys.* **30** 2521
Lou S-y 1990 *J. Phys. A: Math. Gen.* **23** L649
Lou S-y, Ruan H-y, Chen D-f and Chen W-z 1991 *J. Phys. A: Math. Gen.* **24** in press
Nucci M C 1988 *J. Phys. A: Math. Gen.* **21** 73
—— 1989 *J. Phys. A: Math. Gen.* **22** 2897
Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1
Weiss J 1983 *J. Math. Phys.* **24** 1405
Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522